Assignment 8 supplement

Proof rules for the Existential Quantifier

Existential Introduction - ∃I

This rule allows the inference of an existentially quantified claim from any instance of it. For example, we can infer ∃xPx from any of the following: Pa, Pb, Pj, Pn, etc. The conclusion of the use of ∃I depends on the same assumptions as the premise of the rule. Keep the following points in mind when using ∃I:

1) In a correct use of ∃I, the existential quantifier must be the main connective of your conclusion. Thus it is not correct to apply ∃I to ~Pa to get ~∃xPx or to Pa→Qb to get ∃xPx→Pb. Instead, if you have ~Pa and you use ∃I you get ∃x~Px and if you have Pa→Qb and you use ∃I you can get either ∃x(Px→Qb) or ∃x(Pa→Qb) depending on which name you want to quantify over.

2) When applying ∃I to sentences where there is more than one occurrence of the name we are replacing, we may replace any number of occurrences of the name by the variable. For example, from Raa we can infer ∃xRxx, ∃xRax or ∃xRxa. If we wished, in the second two cases we could use ∃I a second time to get ∃y∃xRyx or ∃y∃xRxy.

3) If a sentence contains several different names, such as Rab, then we can apply ∃I to get rid of either the 'a' or the 'b', but we cannot replace both the 'a' and the 'b' by the same variable. Thus from Rab we can infer ∃xRxb or ∃xRax but it is incorrect to infer ∃xRxx since we would have to replace both an 'a' and a 'b' with the same variable.

Existential Elimination - ∃E

This rule allows says that if we can infer a sentence from an arbitrary instance of an existentially quantified claim, then we can infer that sentence from the existentially quantified sentence. In other words, if X follows from Fa and Fa is an arbitrary instance of ∃xFx, then X follows from ∃xFx alone.

To see why this is correct, think of the existential claim – say ∃xFx – as a long disjunction Fa v Fb v Fc v … If we select an arbitrary instance, say Fa, and show that some sentence X follows, then it would follow from each of the disjuncts since Fa was completely arbitrary. Thus ∃E represents an argument by cases where there is only one case. This is acceptable, since if it is an arbitrary case, it represents all possible cases at the same time.

Formally, the rule can be described this way (following the book on page 83):

Given a sentence (at line m) and an assumption (at line i) that is an instance of some existentially quantified sentence (at line k), conclude that sentence again. The new sentence (m+1) depends on all of the assumptions that (m) depended on plus the assumptions that the existential sentence (k) depended on <u>minus</u> the assumption (i).

The conditions that ensure that line (i) was really an arbitrary instance of line (k) are:
1) the instantial name at line (i) that replaced the existentially quantified variable in line (k) must not have appeared in the initial quantified sentence (k) and must not appear in the new line (m). Also, this name cannot appear in any of the assumptions that line (m) depends on (except of course in our instance (i)).

Here is a simple argument which uses ∃E:

1	(1) ∃xFx	A
2	(2) Fa	A
2	(3) ∃yFy	2 ∃I
1	(4) ∃yFy	1,3 ∃E (2)

Here line 3 follows from line 2. But Fa is an arbitrary instance of line 1. In other words, if line 2 used a different name, say it was Fb or Fj, I could have still gotten line 3. In this sense, 'a' was completely arbitrary. So line 3 follows from line 1 regardless of which object is F. Thus line 4 is correct.

Officially to make sure that line 4 is correct I should make sure that line 2 really is an instance of line 1 and that it is an arbitrary instance. To make sure of that, look at the name I put in for the variable – here it is 'a'. 'a' must not occur in any of the sentences that 3 depends on (except for where I put it in – namely line 2). Since this is the only line 3 depends on, this part is satisfied. Also, I must have gotten rid of all of the 'a's. In other words, 'a' can't occur in line 4 either.

<u>Strategy and Examples:</u>

To prove an existentially quantified claim, it would suffice to prove a particular instance of it and then use ∃I. However, existentials are like disjunctions and very often you will not be able to prove any particular instance or sometimes even if you can, you can't tell which instance until later in the proof. Of course you should keep your eyes open for this possibility, but ordinarily, you should not pick an instance of it and try to prove that instance. Instead, just work your way down with what you have.

To use an existentially quantified sentence, we need to use the ∃E rule. To ensure that we can do this, assume an arbitrary instance of that sentence. To ensure that the instance is arbitrary, choose a name that does not occur previously in the proof (or in a goal that you are aiming for). When you are getting used to using the ∃E rule, make sure that when you do use it you check to make sure all the conditions are met.

EXAMPLE 1 ∃xPx ├ ∃x(Px v Qx)

Step 1. To use premise one, I will set up the	1	(1) $\exists xPx$	A
use of $\exists$E by assuming an arbitrary instance	2	(2) Pa	A
of it. Here there are no names anywhere in the			
my proof yet, so any name will do. Now I will		$\exists x(Px \vee Qx)$	new goal
try to prove my goal so that I can repeat it using $\exists$E		$\exists x(Px \vee Qx)$	$\exists$E

Step 2. Now I can prove my new goal by proving	1	(1) $\exists xPx$	A
an instance of it. Once I get to line 4 I should	2	(2) Pa	A
check to make sure the conditions are ready for	2	(3) $Pa \vee Qa$	2 vI
a proper use of $\exists$E – as long as the sentence I have	2	(4) $\exists x(Px \vee Qx)$	3 $\exists$I
no longer contains the name ‘a’, the strategy I used	1	(5) $\exists x(Px \vee Qx)$	1,4 $\exists$E(2)
should guarantee that the conditions are met.			

EXAMPLE 2 $\exists xRxx \vdash \exists x\exists yRxy$

Step 1. To use $\exists xRxx$ I should assume an	1	(1) $\exists xRxx$	A
arbitrary instance of it to set up the use of $\exists$E.	2	(2) Raa	A
Since as of yet there are no names anywhere in			
my proof, any name will do. Now I will prove		$\exists x\exists yRxy$	new goal
my goal in a way that allows me to use $\exists$E.		$\exists x\exists yRxy$	$\exists$E

Step 2. Now I can easily prove an instance of	1	(1) $\exists xRxx$	A
my new goal by using $\exists$I twice. Remember that	2	(2) Raa	A
while $\exists$E has several conditions on it for its use,	2	(3) $\exists yRay$	2 $\exists$I
$\exists$I is a ‘nice’ rule in that you can get rid of names	2	(4) $\exists x\exists yRxy$	3 $\exists$I
pretty much in the way that you want.	1	(5) $\exists x\exists yRxy$	1,4 $\exists$E(2)
Now I check to make sure that line 4 meets the			
appropriate conditions to use $\exists$E and it does.			

EXAMPLE 3 $\forall x(Cx \rightarrow Dx), \exists xCx \vdash \exists xDx$

Step 1. To use line 2, we assume an arbitrary	1	(1) $\forall x(Cx \rightarrow Dx)$	A
instance of it to set up the use of $\exists$E. Then I will	2	(2) $\exists xCx$	A
try to prove my goal.	3	(3) Ca	A
		$\exists xDx$	new goal
		$\exists xDx$	$\exists$E

Step 2. Now it is quite clear how I can use premise 1. I will use $\forall$E to plug ‘a’ into it. Then $\rightarrow$E allows us to get an instance of our goal. When we get to line 6, we should check to make sure that all of the conditions are met for an $\exists$E. Notice that at no earlier time could we have used $\exists$E since our line always contained ‘a’.

1	(1) $\forall x(Cx \rightarrow Dx)$	A
2	(2) $\exists xCx$	A
3	(3) Ca	A
1	(4) $Ca \rightarrow Da$	1 $\forall$E
1,3	(5) Da	3,4 $\rightarrow$E
1,3	(6) $\exists xDx$	5 $\exists$I
1,2	(7) $\exists xDx$	2,6 $\exists$E(3)